Classical phase space singularities and quantization

Johannes Huebschmann *
Université des Sciences et Technologies de Lille
UFR de Mathématiques, CNRS-UMR 8524
59655 VILLENEUVE D'ASCQ, Cédex, France
and
Institute for Theoretical Physics
Universität Leipzig
04109 LEIPZIG, Germany
Johannes.Huebschmann@math.univ-lille1.fr

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Abstract

Simple classical mechanical systems and solution spaces of classical field theories involve singularities. In certain situations these singularities can be understood in terms of the stratified Kähler spaces. We give an overview of a research program whose aim is to develop a holomorphic quantization procedure on stratified Kähler spaces.

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1 Quantum theory and classical singularities

According to Dirac, the *correspondence* between a classical theory and its quantum counterpart should be based on an analogy between their mathematical structures. An interesting issue is then that of the role of singularities in quantum problems. Singularities are known to arise in classical phase spaces. For example, in the hamiltonian picture of a theory, reduction modulo gauge symmetries leads in general to singularities on the classical level. Thus we are running into the question what the significance of singularities on the quantum side could be. Can we ignore them, or is there a quantum structure which has the classical singularities as its shadow? As far as know, one of the first papers in this topic is that of Emmrich and Römer [7]. This paper indicates that wave functions may "congregate" near a *singular* point, which goes counter to the sometimes quoted statement that *singular points in a quantum problem are a set of measure zero so cannot possibly be important.* In a similar vein, Asorey et al observed that vacuum nodes correspond to the chiral gauge orbits of reducible gauge fields with non-trivial magnetic monopole components [3]. It is also noteworthy that in classical mechanics and in classical field theories singularities in the solution spaces are the *rule rather than the exception*. This is in particular true for Yang-Mills theories and for Einstein's gravitational theory where singularities occur even at some of the most interesting and physically relevant solutions, namely at the symmetric ones. It is still not understood what role these singularities might have in quantum gravity. See, for example, Arms, Marsden and Moncrief [1], [2] and the literature there.

2 An example of a classical phase space singularity

In $\mathbb{R}^3$ with coordinates x, y, r, consider the semicone N given by the equation $x^2+y^2=r^2$ and the inequality $r \geq 0$. We refer to this semicone as the *exotic* plane with a single vertex. The semicone N is the classical reduced *phase space* of a single particle moving in ordinary affine space of dimension ≥ 2 with angular momentum zero. This claim will actually be justified in Section 6 below. The reduced Poisson algebra $(C^\infty N, \{\,\cdot\,,\,\cdot\,\})$ may be described in the following fashion: Let x and y be the ordinary coordinate functions in the plane, and consider the algebra $C^\infty N$ of smooth functions in the variables x, y, r subject to the relation $x^2+y^2=r^2$. Define the Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$

on this algebra by

$$\{x,y\} = 2r,\ \{x,r\} = 2y,\ \{y,r\} = -2x,$$

and endow N with the complex structure having $z = x + iy$ as holomorphic coordinate. The Poisson bracket is then *defined at the vertex* we well, away from the vertex the Poisson structure is an ordinary *symplectic* Poisson structure, and the complex structure does *not* "see" the vertex. At the vertex, the radius function r is *not* a smooth function of the variables x and y. Thus the vertex is a singular point for the Poisson structure whereas it is *not* a singular point for the complex analytic structure. The Poisson and complex analytic structures combine to a "stratified Kähler structure". We will explain shortly what this means.

3 Stratified Kähler spaces

In the presence of singularities, restricting quantization to a smooth open dense stratum, sometimes referred to as "top stratum", can result in a loss of information and may in fact lead to *inconsistent* results. To develop a satisfactory notion of Kähler quantization in the presence of singularities, on the classical level, we isolated a notion of "Kähler space with singularities"; we refer to such a space as a *stratified Kähler space*. Ordinary *Kähler quantization* may then be extended to a *quantization scheme over stratified Kähler spaces*.

We will now explain the concept of a *complex analytic stratified Kähler space*. In [15] we introduced a notion of stratified Kähler space which includes that of complex analytic stratified Kähler space. For the sake of clarity, we maintain the distinction between a (possibly more general) stratified Kähler space and a complex analytic one. We do not know whether the two notions really differ. For the present paper, the notion of complex analytic stratified Kähler space suffices.

Let N be a stratified space. We recall first that a *stratified symplectic structure* on N is a Poisson algebra $(C^\infty N, \{\,\cdot\,,\,\cdot\,\})$ of continuous functions on N having the following properties:
(1) Each stratum is an ordinary smooth manifold, and the restriction map from $C^\infty N$ to the algebra of continuous functions on that stratum goes into the algebra of ordinary smooth functions on the stratum.
(2) Each stratum carries a symplectic structure, and the restriction map from $C^\infty N$ to the algebra of smooth functions on the stratum is a morphism of Poisson algebras, where the stratum is endowed with the ordinary smooth symplectic Poisson structure.

A *stratified symplectic space* is a stratified space together with a stratified symplectic structure. The functions in $C^\infty N$ are not necessarily ordinary smooth functions. In the special case where restriction of the functions in $C^\infty N$ to any stratum yields the compactly supported functions on that stratum, the Poisson algebra $(C^\infty N, \{\,\cdot\,,\,\cdot\,\})$ actually induces the symplectic Poisson structure on each stratum.

Next we recall that a *complex analytic space* (in the sense of GRAUERT) is a topological space X, together with a sheaf of rings $\mathcal{O}_X$, having the following property: The space X can be covered by open sets Y, each of which embeds into the open polydisc $U = \{\mathbf{z} = (z_1, \ldots, z_n); |\mathbf{z}| < 1\}$ in some $\mathbb{C}^n$ (the dimension n may vary as U varies) as the zero set of a finite system of holomorphic functions $f_1, \ldots, f_q$ defined on U, such that the restriction $\mathcal{O}_Y$ of the sheaf $\mathcal{O}_X$ to Y is isomorphic as a sheaf to the quotient sheaf $\mathcal{O}_U/(f_1, \ldots, f_q)$; here $\mathcal{O}_U$ is the sheaf of germs of holomorphic functions on U. The sheaf $\mathcal{O}_X$ is then referred to as the *sheaf of holomorphic functions on* X. See [8] for a development of the general theory of complex analytic spaces.

Definition 3.1 *A complex analytic* stratified Kähler space *consists of a stratified space N, together with*
(i) *a stratified symplectic structure $(C^\infty N, \{\,\cdot\,,\,\cdot\,\})$ having the given stratification of N as its underlying stratification, and with*
(ii) *a complex analytic structure on N which is compatible with the stratified symplectic structure.*

The two structures being *compatible* means the following:
(i) Each stratum is a complex analytic subspace, and the complex analytic structure, restricted to the stratum, turns that stratum into an ordinary complex manifold; in particular, the stratification of N is a refinement of the complex analytic stratification.
(ii) For each point q of N and each holomorphic function f defined on an open neighborhood U of q, there is an open neighborhood V of q with $V \subset U$ such that, on V, f is the restriction of a function in $C^\infty(N, \mathbb{C}) = C^\infty(N) \otimes \mathbb{C}$.
(iii) On each stratum, the symplectic structure combines with the complex analytic structure to a Kähler structure.

EXAMPLE 1: The *exotic plane*, endowed with the structure explained in Section 2 above, is a stratified Kähler space. Here the radius function r is *not* an ordinary smooth function of the variables x and y. Thus the stratified symplectic structure cannot be given in terms of ordinary smooth functions of the variables x and y.

This example generalizes to an entire class of examples: The *closure of a holomorphic nilpotent orbit* (in a hermitian Lie algebra) inherits a complex

analytic stratified Kähler structure [15]. Angular momentum zero reduced spaces are special cases thereof; see Section 6 below for details.

Projectivization of the closure of a holomorphic nilpotent orbits yields what we call an *exotic projective variety*. This includes complex quadrics, Severi and Scorza varieties and their *secant* varieties [15], [17]. In physics, spaces of this kind arise as reduced classical phase spaces for systems of harmonic oscillators with zero angular momentum and constant energy. We shall explain some of the details in Section 6 below.

Example 2: A moduli space of semistable holomorphic vector bundles or, more generally, a moduli space of semistable principal bundles on a non-singular complex projective curve carries a complex analytic stratified Kähler structure [15]. As a space, that is, the complex analytic structure being ignored, such a moduli space arises as the moduli space of homomorphisms or more generally twisted homomorphisms from the fundamental group of a (real) surface to a compact connected Lie group as well. In conformal field theory, moduli spaces of this kind occur as spaces of *conformal blocks*. The construction of the moduli spaces as complex projective varieties goes back to [24] and [28]; see [29] for an exposition of the general theory. Atiyah and Bott [5] initiated another approach to the study of these moduli spaces by identifying them with moduli spaces of projectively flat constant central curvature connections on principal bundles over Riemann surfaces, which they analyzed by methods of gauge theory. In particular, by applying the method of symplectic reduction to the action of the infinite-dimensional group of gauge transformations on the infinite-dimensional symplectic manifold of all connections on a principal bundle, they showed that an invariant inner product on the Lie algebra of the Lie group in question induces a natural symplectic structure on a certain smooth open and dense stratum which, together with the complex analytic structure determined by the complex structure on the Riemann surface (complex curve), turns that stratum into an ordinary Kähler manifold. In certain cases, the open and dense stratum is the entire space, which is then a compact Kähler manifold. This infinite-dimensional approach to moduli spaces of the kind under discussion has roots in quantum field theory. Thereafter a finite-dimensional construction of such a moduli space as a symplectic quotient arising from an ordinary finite-dimensional Hamiltonian G-space for a compact Lie group G was developed; see [12], [13] and the literature there. This construction exhibits the moduli space as a *compact* stratified symplectic space. The stratified symplectic structure, in turn, combines with the complex analytic structure determined by the complex structure on the Riemann surface to a complex analytic stratified Kähler structure. This structure includes the Kähler manifold structure on the open and dense stratum; indeed it compactifies this Kähler manifold to

a *compact* complex analytic stratified Kähler space.

An important special case is that of the moduli space of semistable rank 2 degree zero vector bundles with trivial determinant on a curve of genus 2. As a space, this moduli space is a copy of ordinary complex projective 3-space, but the stratified symplectic structure involves more functions than just ordinary smooth functions. In this moduli space, as a complex analytic subspace, the complement of the space of stable vector bundles is a *Kummer surface*, associated with the Jacobian of the curve. Endowed with the additional structure of a complex analytic stratified Kähler space, the Kummer surface is another example of an *exotic* projective variety. See [11]–[13], [15], and the literature there.

Any ordinary Kähler manifold is plainly a complex analytic stratified Kähler space. This kind of example generalizes in the following fashion: For a Lie group K, we will denote its Lie algebra by $\mathfrak{k}$ and the dual thereof by $\mathfrak{k}^*$. The next result says that, roughly speaking, Kähler reduction, applied to an ordinary Kähler manifold, yields a complex analytic stratified Kähler structure on the reduced space.

Theorem 3.2 ([15]) *Let N be a Kähler manifold, acted upon holomorphically by a complex Lie group G such that the action, restricted to a compact real form K of G, preserves the Kähler structure and is hamiltonian, with momentum mapping $\mu\colon N \to \mathfrak{k}^*$. Then the reduced space $N_0 = \mu^{-1}(0)\big/K$ inherits a complex analytic stratified Kähler structure.*

For intelligibility, we explain briefly how the structure on the reduced space N_0 arises. Details may be found in [15]: Consider the algebra $C^\infty(N)^K$ of smooth K-invariant functions on N, and let I^K be the ideal of functions in $C^\infty(N)^K$ that vanish on the zero locus $\mu^{-1}(0)$. Define $C^\infty(N_0)$ to be the quotient algebra $C^\infty(N)^K\big/I^K$. This is an algebra of continuous functions on N_0 in an obvious fashion. The ordinary smooth symplectic Poisson structure $\{\,\cdot\,,\,\cdot\,\}$ on $C^\infty(N)$ is K-invariant and hence induces a Poisson structure on the algebra $C^\infty(N)^K$ of smooth K-invariant functions on N. Furthermore, NOETHER's theorem entails that the ideal I^K is a Poisson ideal, that is to say, given $f \in C^\infty(N_0)^K$ and $h \in I^K$, the function $\{f, h\}$ is in I^K as well. Consequently the Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ descends to a Poisson bracket $\{\,\cdot\,,\,\cdot\,\}_0$ on $C^\infty(N_0)$. Relative to the orbit type stratification, the Poisson algebra $(C^\infty N_0, \{\,\cdot\,,\,\cdot\,\}_0)$ turns N_0 into a stratified symplectic space.

The inclusion of $\mu^{-1}(0)$ into N passes to a homeomorphism from N_0 onto the categorical G-quotient $N\big/\!\big/G$ of N in the category of complex analytic varieties. The stratified symplectic structure combines with the complex analytic structure on $N\big/\!\big/G$ to a stratified Kähler structure. When N is com-

plex algebraic, the complex algebraic G-quotient coincides with the complex analytic G-quotient.

Thus we see that, in view of Theorem 3.2, examples of stratified Kähler spaces abound.

Example 3: Adjoint quotients of complex reductive Lie groups; see [19] for details: Let G be a compact Lie group, and let $G^{\mathbb{C}}$ be its complexification; any complex reductive Lie group is of this kind. Let $\mathfrak{g}$ be the Lie algebra of G. Recall that the polar map from $G\times\mathfrak{g}$ to $G^{\mathbb{C}}$, which is given by the assignment to $(x, Y) \in G \times \mathfrak{g}$ of $x\exp(iY) \in G^{\mathbb{C}}$, is a G-biinvariant diffeomorphism. We endow the Lie algebra $\mathfrak{g}$ with an invariant (positive definite) inner product; by means of this inner product, we identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$ and, furthermore, the total space $\mathrm{T}G$ of the tangent bundle of G with the total space T^*G of the cotangent bundle of G. The composite

$$\mathrm{T}^*G \longrightarrow G \times \mathfrak{g} \longrightarrow G^{\mathbb{C}}$$

of left translation with the polar map is a diffeomorphism, and it may be shown that the resulting complex structure on T^*G combines with the cotangent bundle symplectic structure to a Kähler structure. Moreover, the action of $G^{\mathbb{C}}$ on itself by conjugation is holomorphic, and the restriction of the action to G is Hamiltonian, with momentum mapping from $G^{\mathbb{C}}$ to $\mathfrak{g}^*$ which, viewed as a map on the isomorphe $G \times \mathfrak{g}$ and with values in $\mathfrak{g}$, amounts to the map

$$\mu\colon G \times \mathfrak{g} \longrightarrow \mathfrak{g}, \quad \mu(x, Y) = \mathrm{Ad}_x Y - Y.$$

By Theorem 3.2, the reduced space $(\mathrm{T}^*G)_0$ inherits a stratified Kähler space structure. In physics, a space of the kind $(\mathrm{T}^*G)_0$ is the *building block* for certain *lattice gauge* theories.

Consider the special case where $G = \mathrm{SU}(n)$, so that $G^{\mathbb{C}} = \mathrm{SL}(n, \mathbb{C})$. We claim that, complex analytically, the reduced space $(\mathrm{T}^*G)_0$ amounts to a copy $\mathbb{C}^{n-1}$ of complex affine $(n-1)$-dimensional space. Indeed, the characters $\sigma_1, \ldots, \sigma_{n-1}$ of the fundamental representations of $\mathrm{SL}(n, \mathbb{C})$ yield a holomorphic map

$$(\sigma_1, \ldots, \sigma_{n-1})\colon \mathrm{SL}(n, \mathbb{C}) \longrightarrow \mathbb{C}^{n-1},$$

referred to in the literature as Steinberg map, such that the induced map from $\mathrm{SL}(n, \mathbb{C})//\mathrm{SL}(n, \mathbb{C})$ to $\mathbb{C}^{n-1}$ is an isomorphism of complex algebraic and hence of complex analytic spaces. Hence, as a complex analytic space, the reduced space $(\mathrm{T}^*\mathrm{SL}(n, \mathbb{C}))_0$ comes down to a copy $\mathbb{C}^{n-1}$ of complex affine $(n-1)$-dimensional space.

Here is another way to understand the situation: A maximal complex torus $T^{\mathbb{C}}$ in $\mathrm{SL}(n, \mathbb{C})$ is given by the complex diagonal matrices in $\mathrm{SL}(n, \mathbb{C})$,

that is, by the complex diagonal $(n \times n)$-matrices having determinant 1. Realize the torus $T^{\mathbb{C}}$ as the subspace of $(\mathbb{C}^*)^n$ which consists of all $(z_1, \ldots, z_n)$ in $(\mathbb{C}^*)^n$ such that $z_1 \ldots z_n = 1$; then the characters $\sigma_1, \ldots, \sigma_{n-1}$, restricted to $T^{\mathbb{C}}$, are the elementary symmetric functions in the variables $z_1, \ldots, z_n$. The Steinberg map restricts to the holomorphic map

$$(\sigma_1, \ldots, \sigma_{n-1})\colon (\mathbb{C}^*)^{n-1} \longrightarrow \mathbb{C}^{n-1}$$

given by the assignment to $\mathbf{z} = (z_1, \ldots, z_n) \in T^{\mathbb{C}}$ of $(\sigma_1(\mathbf{z}), \ldots, \sigma_{n-1}(\mathbf{z}))$. Since, for a general complex Lie group of the kind $G^{\mathbb{C}}$, every conjugacy class has in its closure a unique semisimple conjugacy class and since semisimple conjugacy classes are parametrized by orbits of the action of the Weyl group on the maximal complex torus $T^{\mathbb{C}}$, we see that the restriction of the Steinberg map to the maximal torus already realizes the complex analytic quotient as the space of orbits relative to the action of the Weyl group on $T^{\mathbb{C}}$, which is here the symmetric group S_n on n letters.

As a *stratified Kähler space*, the adjoint quotient is considerably more complicated. We will restrict attention to the even more special case where $n = 2$, so that $G = \mathrm{SU}(2)$ and $G^{\mathbb{C}} = \mathrm{SL}(2, \mathbb{C})$. Then

$$T = \left\{ \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{bmatrix}, |\zeta| = 1 \right\}$$

is a maximal torus in G, and

$$\mathbb{C}^* \cong T^{\mathbb{C}} = \left\{ \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{bmatrix}, \zeta \neq 0 \right\}$$

is a maximal torus in $G^{\mathbb{C}} = \mathrm{SL}(2, \mathbb{C})$. In view of the above discussion, complex analytically, the adjoint quotient $G^{\mathbb{C}}/\!/G^{\mathbb{C}}$ amounts to the space $T^{\mathbb{C}}\big/(\mathbb{Z}/2) \cong \mathbb{C}$ of orbits relative to the action of the Weyl group S_2 on $\mathbb{C}^* \cong T^{\mathbb{C}}$, and this orbit space is realized as the target of the holomorphic map

$$f\colon \mathbb{C}^* \longrightarrow \mathbb{C},\ f(z) = z + z^{-1}.$$

Thus $Z = z + z^{-1}$ may be taken as a holomorphic coordinate on the adjoint quotient.

On the other hand, the real structure is encapsulated by the *real stratified symplectic Poisson structure* $(C^\infty(\mathrm{T}^*G)_0, \{\,\cdot\,,\,\cdot\,\})$ which, in turn, involves the three functions

$$X = x + \frac{x}{r^2},\ Y = y - \frac{y}{r^2},\ \tau = \frac{y^2}{r^2},$$

where $z = x + iy$, $Z = X + iY$, and $x^2 + y^2 = r^2$. The resulting smooth structure $C^\infty(\mathrm{T}^*G)_0$ on the quotient

$$G^{\mathbb{C}}//G^{\mathbb{C}} \cong (\mathrm{T}^*G)_0 \cong \mathbb{C}$$

is the algebra of smooth functions in the variables X, Y, τ, subject to the relation

$$Y^2 = (X^2 + Y^2 + 4(\tau - 1))\tau.$$

Moreover, the Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ on $C^\infty(\mathrm{T}^*G)_0$ is determined by

$$\{X, Y\} = X^2 + Y^2 + 4(2\tau - 1).$$

The resulting complex analytic stratified Kähler structure is *singular* at the points -2 and 2, that is, the Poisson structure vanishes at these points; furthermore, at these two points, the function τ is *not* an ordinary smooth function of the variables X and Y. Away from these two points, the Poisson structure is symplectic. We refer to this adjoint quotient as the *exotic plane with two vertices*. It is perhaps worthwhile noting that this complex analytic stratified Kähler space also arises as the reduced classical phase space of a spherical pendulum constrained to move with angular momentum zero, so that it moves in a plane. See [19] for details.

4 Correspondence principle and Lie-Rinehart algebras

To make sense of the *correspondence principle* in certain *singular* situations, one needs a tool which, for the stratified symplectic Poisson algebra on a stratified symplectic space, serves as a *replacement* for the tangent bundle of a smooth symplectic manifold. This replacement is provided for by an appropriate *Lie-Rinehart algebra*. This Lie-Rinehart algebra yields in particular a satisfactory generalization of the Lie algebra of smooth vector fields in the smooth case. This enables us to put *flesh on the bones of Dirac's correspondence principle in certain singular situations*.

A *Lie-Rinehart algebra* consists of a commutative algebra and a Lie algebra with additional structure which generalizes the mutual structure of interaction between the algebra of smooth functions and the Lie algebra of smooth vector fields on a smooth manifold. More precisely:

Definition 4.1 *A* Lie-Rinehart *algebra consists of a commutative algebra A and a Lie-algebra L such that L acts on A by derivations and that L has an*

A-module structure, and these are required to satisfy

$$[\alpha, a\beta] = \alpha(a)\beta + a[\alpha, \beta],$$
$$(a\alpha)(b) = a(\alpha(b)),$$

where $a, b \in A$ and $\alpha, \beta \in L$.

We will now explain briefly the Lie-Rinehart algebra associated with a Poisson algebra; more details may be found in [9], [10], and [18]. Thus, let $(A, \{\,\cdot\,,\,\cdot\,\})$ be a Poisson algebra. Let D_A be the A-module of formal differentials of A the elements of which we write as du, for $u \in A$. For $u, v \in A$, the association

$$(du, dv) \longrightarrow \pi(du, dv) = \{u, v\}$$

yields an A-valued A-bilinear skew-symmetric 2-form $\pi = \pi_{\{\,\cdot\,,\,\cdot\,\}}$ on D_A, referred to as the *Poisson* 2-*form* associated with the Poisson structure $\{\,\cdot\,,\,\cdot\,\}$. The adjoint

$$\pi^\sharp\colon D_A \longrightarrow \mathrm{Der}(A) = \mathrm{Hom}_A(D_A, A)$$

of π is a morphism of A-modules, and the formula

$$[adu, bdv] = a\{u, b\}dv + b\{a, v\}du + abd\{u, v\}$$

yields a Lie bracket $[\,\cdot\,,\,\cdot\,]$ on D_A.

Theorem 4.2 ([9]) *The A-module structure on D_A, the bracket $[\cdot, \cdot]$, and the morphism $\pi^\sharp$ of A-modules turn the pair (A, D_A) into a Lie-Rinehart algebra.*

We will write the resulting Lie-Rinehart algebra as $(A, D_{\{\,\cdot\,,\,\cdot\,\}})$. For intelligibility we recall that, given a Lie-Rinehart algebra (A, L), the Lie algebra L together with the additional A-module structure on L and L-module structure on A is referred to as an $(\mathbb{R}, A)$-*Lie algebra*. Thus $D_{\{\,\cdot\,,\,\cdot\,\}}$ is an $(\mathbb{R}, A)$-Lie algebra.

When the Poisson algebra A is the algebra of smooth functions $C^\infty(N)$ on a symplectic manifold N, endowed with the ordinary symplectic Poisson structure, the A-dual $\mathrm{Der}(A) = \mathrm{Hom}_A(D_A, A)$ of D_A amounts to the A-module $\mathrm{Vect}(N)$ of smooth vector fields, and

$$(\pi^\sharp, \mathrm{Id})\colon (D_A, A) \longrightarrow (\mathrm{Vect}(N), C^\infty(N)) \tag{4.1}$$

is a morphism of Lie-Rinehart algebras, where $(\mathrm{Vect}(N), C^\infty(N))$ carries its ordinary Lie-Rinehart structure. The A-module morphism $\pi^\sharp$ is plainly

surjective, and the kernel consists of those formal differentials which "vanish at each point of" N.

We return to our general Poisson algebra $(A, \{\,\cdot\,,\,\cdot\,\})$. The Poisson 2-form $\pi_{\{\cdot,\cdot\}}$ determines an *extension*

$$0 \longrightarrow A \longrightarrow \overline{L}_{\{\cdot,\cdot\}} \longrightarrow D_{\{\cdot,\cdot\}} \longrightarrow 0 \tag{4.2}$$

of $(\mathbb{R}, A)$-Lie algebras which is central as an extension of ordinary Lie algebras; in particular, on the kernel A, the Lie bracket is trivial. Moreover, as A-modules,

$$\overline{L}_{\{\cdot,\cdot\}} = A \oplus D_{\{\cdot,\cdot\}}, \tag{4.3}$$

and the Lie bracket on $\overline{L}_{\{\cdot,\cdot\}}$ is given by

$$[(a, du), (b, dv)] = (\{u, b\} + \{a, v\} - \{u, v\}, d\{u, v\}), \quad a, b, u, v \in A. \tag{4.4}$$

Here we have written "$\overline{L}$" rather than simply L to indicate that the extension (4.2) represents the *negative* of the class of $\pi_{\{\cdot,\cdot\}}$ in Poisson cohomology $\mathrm{H}^2_{\mathrm{Poisson}}(A, A)$, cf. [9]. When $(A, \{\,\cdot\,,\,\cdot\,\})$ is the smooth symplectic Poisson algebra of an ordinary smooth symplectic manifold, (perhaps) up to sign, the class of $\pi_{\{\cdot,\cdot\}}$ comes essentially down to the cohomology class represented by the symplectic structure.

The following concept has been introduced in [10].

Definition 4.3 *Given an $(A \otimes \mathbb{C})$-module M, we refer to an $(A, \overline{L}_{\{\cdot,\cdot\}})$-module structure*

$$\chi\colon \overline{L}_{\{\cdot,\cdot\}} \longrightarrow \mathrm{End}_{\mathbb{R}}(M) \tag{4.5}$$

on M as a prequantum module structure for $(A, \{\,\cdot\,,\,\cdot\,\})$ *provided*
(i) *the values of χ lie in* $\mathrm{End}_{\mathbb{C}}(M)$*, that is to say, for $a \in A$ and $\alpha \in D_{\{\cdot,\cdot\}}$, the operators $\chi(a, \alpha)$ are complex linear transformations, and*
(ii) *for every $a \in A$, with reference to the decomposition* (4.3)*, we have*

$$\chi(a, 0) = i\, a\, \mathrm{Id}_M. \tag{4.6}$$

A pair (M, χ) consisting of an $(A \otimes \mathbb{C})$-module M and a prequantum module structure will henceforth be referred to as a prequantum module *(for $(A, \{\,\cdot\,,\,\cdot\,\})$.*

Prequantization now proceeds in the following fashion, cf. [9]: The assignment to $a \in A$ of $(a, da) \in \overline{L}_{\{\cdot,\cdot\}}$ yields a morphism ι of real Lie algebras from A to $\overline{L}_{\{\cdot,\cdot\}}$; thus, for any prequantum module (M, χ), the composite of ι with $-i\chi$ is a representation $a \mapsto \widehat{a}$ of the A underlying real Lie algebra having M, viewed as a complex vector space, as its representation space;

this is a representation by $\mathbb{C}$-linear operators so that any constant acts by multiplication, that is, for any real number r, viewed as a member of A,

$$\widehat{r} = r \,\mathrm{Id} \tag{4.7}$$

and so that, for $a, b \in A$,

$$\widehat{\{a,b\}} = i\,[\widehat{a}, \widehat{b}] \qquad \text{(the Dirac condition).} \tag{4.8}$$

More explicitly, these operators are given by the formula

$$\widehat{a}(x) = \frac{1}{i}\chi(0, da)(x) + ax, \quad a \in A,\ x \in M. \tag{4.9}$$

In this fashion, prequantization, that is to say, the first step in the realization of the correspondence principle in one direction, can be made precise in certain singular situations.

When $(A, \{\,\cdot\,,\,\cdot\,\})$ is the Poisson algebra of smooth functions on an ordinary smooth sympletic manifold, this prequantization factors through the morphism (4.1) of Lie-Rinehart algebras in such a way that, on the target, the construction comes down to the ordinary prequantization construction.

REMARK. In the physics literature, Lie-Rinehart algebras were explored in a paper by KASTLER AND STORA under the name *Lie-Cartan pairs* [20].

5 Quantization on stratified Kähler spaces

In the paper [16] we have shown that the *holomorphic* quantization scheme may be extended to complex analytic stratified Kähler spaces. We recall the main steps:

1) The notion of ordinary Kähler polarization generalizes to that of *stratified Kähler polarization*. This concept is defined in terms of the Lie-Rinehart algebra associated with the stratified symplectic Poisson structure; it specifies *polarizations on the strata* and, moreover, encapsulates the *mutual positions of polarizations on the strata*.

Under the circumstances of Theorem 3.2, *symplectic reduction carries a Kähler polarization preserved by the symmetries into a stratified Kähler polarization*.

2) The notion of prequantum bundle generalizes to that of *stratified prequantum module*. Given a complex analytic stratified Kähler space, a stratified prequantum module is, roughly speaking, a system of prequantum modules in the sense of Definition 4.3, one for the closure of each structum, together with appropriate morphisms among them which reflect the stratification.

3) The notion of quantum Hilbert space generalizes to that of *costratified quantum Hilbert space* in such a way that the costratified structure reflects the stratification on the classical level.
4) The main result says that $[Q, R] = 0$, that is, quantization commutes with reduction [16]:

Theorem 5.1 *Under the circumstances of* Theorem 3.2*, suppose that the Kähler manifold is quantizable (that is, suppose that the cohomology class of the Kähler form is integral). When a suitable additional condition is satisfied, reduction after quantization coincides with quantization after reduction in the sense that not only the reduced and unreduced quantum phase spaces correspond but the (invariant) unreduced and reduced quantum observables as well.*

What is referred to here as 'suitable additional condition' is a condition on the behaviour of the gradient flow. For example, when the Kähler manifold is compact, the condition will automatically be satisfied.

On the reduced level, the resulting classical phase space involves in general singularities and is a complex analytic stratified Kähler space; the appropriate quantum phase space is then a costratified Hilbert space.

In collaboration with M. Schmidt and G. Rudolph, we plan to explore the question whether, under certain circumstances, the lower strata have physical significance. This may be seen as another version of the question spelled out above concerning the *quantum structure which might have the classical singularities as its shadow*.

6 An illustration

Let s and ℓ be non-zero natural numbers. The unreduced classical momentum phase space of ℓ particles in $\mathbb{R}^s$ is real affine space of real dimension $2s\ell$. Identify this space with the vector space $(\mathbb{R}^{2s})^{\times \ell}$ as usual, endow $\mathbb{R}^s$ with the standard inner product, $\mathbb{R}^{2\ell}$ with the standard symplectic structure, and thereafter $(\mathbb{R}^{2s})^{\times \ell}$ with the obvious induced inner product and symplectic structure. The isometry group of the inner product on $\mathbb{R}^s$ is the orthogonal group $\mathrm{O}(s, \mathbb{R})$, the group of linear transformations preserving the symplectic structure on $\mathbb{R}^{2\ell}$ is the symplectic group $\mathrm{Sp}(\ell, \mathbb{R})$, and the actions extend to linear $\mathrm{O}(s, \mathbb{R})$- and $\mathrm{Sp}(\ell, \mathbb{R})$-actions on $(\mathbb{R}^{2s})^{\times \ell}$ in an obvious manner. As usual, denote the Lie algebras of $\mathrm{O}(s, \mathbb{R})$ and $\mathrm{Sp}(\ell, \mathbb{R})$ by $\mathfrak{so}(s, \mathbb{R})$ and $\mathfrak{sp}(\ell, \mathbb{R})$, respectively.

The $\mathrm{O}(s, \mathbb{R})$- and $\mathrm{Sp}(\ell, \mathbb{R})$-actions on $(\mathbb{R}^{2s})^{\times \ell}$ are hamiltonian. To spell out the $\mathrm{O}(s, \mathbb{R})$-momentum mapping having the value zero at the origin,

identify $\mathfrak{so}(s,\mathbb{R})$ with its dual $\mathfrak{so}(s,\mathbb{R})^*$ by interpreting $a \in \mathfrak{so}(s,\mathbb{R})$ as the linear functional on $\mathfrak{so}(s,\mathbb{R})$ which assigns $\mathrm{tr}(a^t x)$ to $x \in \mathfrak{so}(s,\mathbb{R})$; here ${}^t x$ refers to the transpose of the matrix x. We note that, for $s \geq 3$,

$$(s-2)\mathrm{tr}(a^t b) = -\beta(a,b),\ \ a,b \in \mathfrak{so}(s,\mathbb{R}),$$

where β is the Killing form of $\mathfrak{so}(s,\mathbb{R})$. Moreover, for a vector $\mathbf{x} \in \mathbb{R}^s$, realized as a column vector, let ${}^t\mathbf{x}$ be its transpose, so that ${}^t\mathbf{x}$ is a row vector. With these preparations out of the way, the *angular momentum mapping*

$$\mu_{\mathrm{O}} \colon (\mathbb{R}^{2s})^{\times \ell} \longrightarrow \mathfrak{so}(s,\mathbb{R})$$

with reference to the origin is given by

$$\mu_{\mathrm{O}}(\mathbf{q}_1,\mathbf{p}_1,\ldots,\mathbf{q}_\ell,\mathbf{p}_\ell) = \mathbf{q}_1{}^t\mathbf{p}_1 - \mathbf{p}_1{}^t\mathbf{q}_1 + \cdots + \mathbf{q}_\ell{}^t\mathbf{p}_\ell - \mathbf{p}_\ell{}^t\mathbf{q}_\ell.$$

Likewise, identify $\mathfrak{sp}(\ell,\mathbb{R})$ with its dual $\mathfrak{sp}(\ell,\mathbb{R})^*$ by interpreting $a \in \mathfrak{sp}(\ell,\mathbb{R})$ as the linear functional on $\mathfrak{sp}(\ell,\mathbb{R})$ which assigns $\frac{1}{2}\mathrm{tr}(ax)$ to $x \in \mathfrak{sp}(\ell,\mathbb{R})$; we remind the reader that the *Killing* form β of $\mathfrak{sp}(\ell,\mathbb{R})$ satisfies the identity

$$\beta(a,b) = 2(\ell+1)\mathrm{tr}(ab)$$

where $a,b \in \mathfrak{sp}(\ell,\mathbb{R})$. In terms of the present identification of $\mathfrak{sp}(\ell,\mathbb{R})$ with its dual, the $\mathrm{Sp}(\ell,\mathbb{R})$-momentum mapping

$$\mu_{\mathrm{Sp}} \colon (\mathbb{R}^{2s})^{\times \ell} \longrightarrow \mathfrak{sp}(\ell,\mathbb{R})$$

having the value zero at the origin is given by the assignment to

$$[\mathbf{q}_1,\mathbf{p}_1,\ldots,\mathbf{q}_\ell,\mathbf{p}_\ell] \in (\mathbb{R}^s \times \mathbb{R}^s)^{\times \ell}$$

of

$$\left[\begin{array}{cc} [\mathbf{q}_j\mathbf{p}_k] & -[\mathbf{q}_j\mathbf{q}_k] \\ [\mathbf{p}_j\mathbf{p}_k] & -[\mathbf{p}_j\mathbf{q}_k] \end{array}\right] \in \mathfrak{sp}(\ell,\mathbb{R}),$$

where $[\mathbf{q}_j\mathbf{p}_k]$ etc. denotes the $(\ell \times \ell)$-matrix having the inner products $\mathbf{q}_j\mathbf{p}_k$ etc. as entries.

Consider the $\mathrm{O}(s,\mathbb{R})$-reduced space

$$N_0 = \mu_{\mathrm{O}}^{-1}(0)/\mathrm{O}(s,\mathbb{R}).$$

The $\mathrm{Sp}(\ell,\mathbb{R})$-momentum mapping μ_{Sp} induces an embedding of the reduced space N_0 into $\mathfrak{sp}(\ell,\mathbb{R})$. We now explain briefly how the image of N_0 in $\mathfrak{sp}(\ell,\mathbb{R})$ may be described. More details may be found in [15], see also [17].

Choose a positive complex structure J on $\mathbb{R}^{2\ell}$ which is compatible with ω in the sense that $\omega(J\mathbf{u}, J\mathbf{v}) = \omega(\mathbf{u}, \mathbf{v})$ for every $\mathbf{u}, \mathbf{v} \in \mathbb{R}^{2\ell}$; here 'positive' means that the associated real inner product $\cdot$ on $\mathbb{R}^{2\ell}$ given by $\mathbf{u} \cdot \mathbf{v} = \omega(\mathbf{u}, J\mathbf{v})$ for $\mathbf{u}, \mathbf{v} \in \mathbb{R}^{2\ell}$ is positive definite. The subgroup of $\mathrm{Sp}(\ell, \mathbb{R})$ which preserves the complex structure J is a maximal compact subgroup of $\mathrm{Sp}(\ell, \mathbb{R})$; relative to a suitable orthonormal basis, this group comes down to a copy of the ordinary unitary group $\mathrm{U}(\ell)$. Furthermore, the complex structure J induces a Cartan decomposition

$$\mathfrak{sp}(\ell, \mathbb{R}) = \mathfrak{u}(\ell) \oplus \mathfrak{p}; \tag{6.1}$$

here $\mathfrak{u}(\ell)$ is the Lie algebra of $\mathrm{U}(\ell)$, $\mathfrak{p}$ decomposes as the direct sum

$$\mathfrak{p} \cong \mathrm{S}^2_{\mathbb{R}}[\mathbb{R}^\ell] \oplus \mathrm{S}^2_{\mathbb{R}}[\mathbb{R}^\ell]$$

of two copies of the real vector space $\mathrm{S}^2_{\mathbb{R}}[\mathbb{R}^\ell]$ of real symmetric $(\ell \times \ell)$-matrices, and the complex structure J induces a complex structure on $\mathrm{S}^2_{\mathbb{R}}[\mathbb{R}^\ell] \oplus \mathrm{S}^2_{\mathbb{R}}[\mathbb{R}^\ell]$ in such a way that the resulting complex vector space is complex linearly isomorphic to the complex vector space $\mathrm{S}^2_{\mathbb{C}}[\mathbb{C}^\ell]$ of complex symmetric $(\ell \times \ell)$-matrices in a canonical fashion. We refer to a nilpotent orbit $\mathcal{O}$ in $\mathfrak{sp}(\ell, \mathbb{R})$ as being *holomorphic* if the orthogonal projection from $\mathfrak{sp}(\ell, \mathbb{R})$ to $\mathrm{S}^2_{\mathbb{C}}[\mathbb{C}^\ell]$, restricted to $\mathcal{O}$, is a diffeomorphism from $\mathcal{O}$ onto its image in $\mathrm{S}^2_{\mathbb{C}}[\mathbb{C}^\ell]$. The diffeomorphism from a holomorphic nilpotent orbit $\mathcal{O}$ onto its image in $\mathrm{S}^2_{\mathbb{C}}[\mathbb{C}^\ell]$ extends to a homeomorphism from the closure $\overline{\mathcal{O}}$ onto its image in $\mathrm{S}^2_{\mathbb{C}}[\mathbb{C}^\ell]$, and the closures of the holomorphic nilpotent orbits constitute an ascending sequence

$$0 \subseteq \overline{\mathcal{O}}_1 \subseteq \cdots \subseteq \overline{\mathcal{O}}_k \subseteq \cdots \subseteq \overline{\mathcal{O}}_\ell \subseteq \mathfrak{sp}(\ell, \mathbb{R}),\ 1 \leq k \leq \ell, \tag{6.2}$$

such that, for $1 \leq k \leq \ell$, the orthogonal projection from $\mathfrak{sp}(\ell, \mathbb{R})$ to $\mathrm{S}^2_{\mathbb{C}}[\mathbb{C}^\ell]$, restricted to $\overline{\mathcal{O}}_k$, is a homeomorphism from $\overline{\mathcal{O}}_k$ onto the space of complex symmetric $(\ell \times \ell)$-matrices of rank at most equal to k; in particular, the orthogonal projection from $\mathfrak{sp}(\ell, \mathbb{R})$ to $\mathrm{S}^2_{\mathbb{C}}[\mathbb{C}^\ell]$, restricted to $\overline{\mathcal{O}}_\ell$, is a homeomorphism from $\overline{\mathcal{O}}_\ell$ onto $\mathrm{S}^2_{\mathbb{C}}[\mathbb{C}^\ell]$. Furthermore, each space of the kind $\overline{\mathcal{O}}_k$ is a *stratified* space, the stratification being given by the decomposition according to the rank of the corresponding complex symmetric $(\ell \times \ell)$-matrices in the homeomorphic image in $\mathrm{S}^2_{\mathbb{C}}[\mathbb{C}^\ell]$.

The Lie bracket of the Lie algebra $\mathfrak{sp}(\ell, \mathbb{R})$ induces a Poisson bracket on the algebra $C^\infty(\mathfrak{sp}(\ell, \mathbb{R})^*)$ of smooth functions on the dual $\mathfrak{sp}(\ell, \mathbb{R})^*$ of $\mathfrak{sp}(\ell, \mathbb{R})$ in a canonical fashion. Via the identification of $\mathfrak{sp}(\ell, \mathbb{R})$ with its dual, the Lie bracket on $\mathfrak{sp}(\ell, \mathbb{R})$ induces a Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ on $C^\infty(\mathfrak{sp}(\ell, \mathbb{R}))$. Indeed, the assignment to $a \in \mathfrak{sp}(\ell, \mathbb{R})$ of the linear function

$$f_a \colon \mathfrak{sp}(\ell, \mathbb{R}) \longrightarrow \mathbb{R}$$

given by $f_a(x) = \frac{1}{2}\mathrm{tr}(ax)$ induces a linear isomorphism

$$\mathfrak{sp}(\ell,\mathbb{R}) \longrightarrow \mathfrak{sp}(\ell,\mathbb{R})^*; \tag{6.3}$$

let

$$[\,\cdot\,,\,\cdot\,]^* \colon \mathfrak{sp}(\ell,\mathbb{R})^* \otimes \mathfrak{sp}(\ell,\mathbb{R})^* \longrightarrow \mathfrak{sp}(\ell,\mathbb{R})^*$$

be the bracket on $\mathfrak{sp}(\ell,\mathbb{R})^*$ induced by the Lie bracket on $\mathfrak{sp}(\ell,\mathbb{R})$. The Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ on the algebra $C^\infty(\mathfrak{sp}(\ell,\mathbb{R}))$ is given by the formula

$$\{f,h\}(x) = [f'(x),h'(x)]^*(x),\ x \in \mathfrak{sp}(\ell,\mathbb{R}).$$

The isomorphism (6.3) induces an embedding of $\mathfrak{sp}(\ell,\mathbb{R})$ into $C^\infty(\mathfrak{sp}(\ell,\mathbb{R}))$, and this embedding is plainly a morphism

$$\delta \colon \mathfrak{sp}(\ell,\mathbb{R}) \longrightarrow C^\infty(\mathfrak{sp}(\ell,\mathbb{R}))$$

of Lie algebras when $C^\infty(\mathfrak{sp}(\ell,\mathbb{R}))$ is viewed as a real Lie algebra via the Poisson bracket. In the literature, a morphism of the kind δ is referred to as a *comomentum* mapping.

Let $\mathcal{O}$ be a holomorphic nilpotent orbit. The embedding of the closure $\overline{\mathcal{O}}$ of $\mathcal{O}$ into $\mathfrak{sp}(\ell,\mathbb{R})$ induces a map from the algebra $C^\infty(\mathfrak{sp}(\ell,\mathbb{R}))$ of ordinary smooth functions on $\mathfrak{sp}(\ell,\mathbb{R})$ to the algebra $C^0(\overline{\mathcal{O}})$ of continuous functions on $\overline{\mathcal{O}}$, and we denote the image of $C^\infty(\mathfrak{sp}(\ell,\mathbb{R}))$ in $C^0(\overline{\mathcal{O}})$ by $C^\infty(\overline{\mathcal{O}})$. By construction, each function in $C^\infty(\overline{\mathcal{O}})$ is the restriction of an ordinary smooth function on the ambient space $\mathfrak{sp}(\ell,\mathbb{R})$. Since each stratum of $\overline{\mathcal{O}}$ is an ordinary smooth closed submanifold of $\mathfrak{sp}(\ell,\mathbb{R})$, the functions in $C^\infty(\overline{\mathcal{O}})$, restricted to a stratum of $\overline{\mathcal{O}}$, are ordinary smooth functions on that stratum. Hence $C^\infty(\overline{\mathcal{O}})$ is a *smooth structure* on $\overline{\mathcal{O}}$. The algebra $C^\infty(\overline{\mathcal{O}})$ is referred to as the algebra of Whitney-smooth functions on $\overline{\mathcal{O}}$, relative to the embedding of $\overline{\mathcal{O}}$ into the affine space $\mathfrak{sp}(\ell,\mathbb{R})$. Under the identification (6.3), the orbit $\mathcal{O}$ passes to a *coadjoint* orbit. Consequently, under the surjection $C^\infty(\mathfrak{sp}(\ell,\mathbb{R})) \to C^\infty(\overline{\mathcal{O}})$, the Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ on the algebra $C^\infty(\mathfrak{sp}(\ell,\mathbb{R}))$ descends to a Poisson bracket on $C^\infty(\overline{\mathcal{O}})$, which we still denote by $\{\,\cdot\,,\,\cdot\,\}$, with a slight abuse of notation. This Poisson algebra turns $\overline{\mathcal{O}}$ into a stratified symplectic space. Combined with the complex analytic structure coming from the projection from $\overline{\mathcal{O}}$ onto the corresponding space of complex symmetric $(\ell\times\ell)$-matrices, in this fashion, the space $\overline{\mathcal{O}}$ acquires a *complex analytic stratified Kähler space* structure. The composite of the above comomentum mapping δ with the projection from $C^\infty(\mathfrak{sp}(\ell,\mathbb{R}))$ to $C^\infty(\overline{\mathcal{O}})$ yields an embedding

$$\delta_{\mathcal{O}} \colon \mathfrak{sp}(\ell,\mathbb{R}) \longrightarrow C^\infty(\overline{\mathcal{O}}) \tag{6.4}$$

which is still a morphism of Lie algebras and therefore a comomentum mapping in the appropriate sense.

The $\mathrm{Sp}(\ell,\mathbb{R})$-momentum mapping induces an embedding of the reduced space N_0 into $\mathfrak{sp}(\ell,\mathbb{R})$ which identifies N_0 with the closure $\overline{\mathcal{O}}_{\min(s,\ell)}$ of the holomorphic nilpotent orbit $\mathcal{O}_{\min(s,\ell)}$ in $\mathfrak{sp}(\ell,\mathbb{R})$. In this fashion, the reduced space N_0 inherits a complex analytic stratified Kähler structure. Since the $\mathrm{Sp}(\ell,\mathbb{R})$-momentum mapping induces an identification of N_0 with $\overline{\mathcal{O}}_s$ for every $s \leq \ell$ in a compatible manner, the ascending sequence (6.2), and in particular the notion of holomorphic nilpotent orbit, is actually independent of the choice of complex structure J on $\mathrm{R}^{2\ell}$. For a single particle, i. e. $\ell = 1$, the description of the reduced space N_0 comes down to that of the semicone given in Section 2 above.

In the case at hand, we will now explain briefly the quantization procedure developed in [16]. Suppose that $s \leq \ell$ (for simplicity), let $m = s\ell$, and endow the affine coordinate ring of $\mathbb{C}^m$, that is, the polynomial algebra $\mathbb{C}[z_1,\dots,z_m]$, with the inner product $\cdot$ given by the standard formula

$$\psi \cdot \psi' = \int \psi\overline{\psi'}\mathrm{e}^{-\frac{\mathbf{z}\overline{\mathbf{z}}}{2}}\varepsilon_m, \quad \varepsilon_m = \frac{\omega^m}{(2\pi)^m m!}, \tag{6.5}$$

where ω refers to the symplectic form on $\mathbb{C}^m$. Furthermore, endow the polynomial algebra $\mathbb{C}[z_1,\dots,z_m]$ with the induced $\mathrm{O}(s,\mathbb{R})$-action. By construction, the affine complex coordinate ring $\mathbb{C}[\overline{\mathcal{O}}_s]$ of $\overline{\mathcal{O}}_s$ is canonically isomorphic to the algebra

$$\mathbb{C}[z_1,\dots,z_m]^{\mathrm{O}(s,\mathbb{R})}$$

of $\mathrm{O}(s,\mathbb{R})$-invariants in $\mathbb{C}[z_1,\dots,z_m]$. The restriction of the inner product $\cdot$ to $\mathbb{C}[\overline{\mathcal{O}}_s]$ turns $\mathbb{C}[\overline{\mathcal{O}}_s]$ into a pre-Hilbert space, and HILBERT space completion yields a HILBERT space which we write as $\widehat{\mathbb{C}}[\overline{\mathcal{O}}_s]$. This is the Hilbert space which arises by *holomorphic quantization* on the complex analytic stratified Kähler space $\overline{\mathcal{O}}_s$; see [16] for details. On this Hilbert space, the elements of the Lie algebra $\mathfrak{u}(\ell)$ of the unitary group $\mathrm{U}(\ell)$ act in an obvious fashion; indeed, the elements of $\mathfrak{u}(\ell)$, viewed as functions in $C^\infty(\overline{\mathcal{O}}_s)$, are classical observables which are directly quantizable, and quantization yields the obvious $\mathfrak{u}(\ell)$-representation on $\mathbb{C}[\overline{\mathcal{O}}_s]$. This construction may be carried out for any $s \leq \ell$ and, for each $s \leq \ell$, the resulting quantizations yields a *costratified Hilbert space* of the kind

$$\mathbb{C} \longleftarrow \widehat{\mathbb{C}}[\overline{\mathcal{O}_1}] \longleftarrow \ldots \longleftarrow \widehat{\mathbb{C}}[\overline{\mathcal{O}_s}]. \tag{6.6}$$

Here each arrow is just a restriction mapping and is actually a morphism of representations for the corresponding quantizable observables, in particular, a morphism of $\mathfrak{u}(\ell)$-representations; each arrow amounts essentially to

an orthogonal projection. Plainly, the costratified structure integrates to a costratified $\mathrm{U}(\ell)$-representation, i. e. to a corresponding system of $\mathrm{U}(\ell)$-representations. The resulting costratified quantum phase space for $\overline{\mathcal{O}_s}$ of the kind (6.6) may be viewed as a *singular* Fock space. This quantum phase space is entirely given in terms of *data on the reduced level*.

Consider the unreduced classical harmonic oscillator energy E which is given by $E = z_1\overline{z}_1 + \cdots + z_m\overline{z}_m$; it quantizes to the Euler operator (quantized harmonic oscillator hamiltonian). For $s \leq \ell$, the reduced classical phase space Q_s of ℓ harmonic oscillators in $\mathbb{R}^s$ with total angular momentum zero and fixed energy value (say) $2k$ fits into an ascending sequence

$$Q_1 \subseteq \cdots \subseteq Q_s \subseteq \cdots \subseteq Q_\ell \cong \mathbb{C}\mathrm{P}^d \tag{6.7}$$

of complex analytic stratified Kähler spaces where

$$\mathbb{C}\mathrm{P}^d = \mathrm{P}(\mathrm{S}^2[\mathbb{C}^\ell]), \quad d = \frac{\ell(\ell+1)}{2} - 1.$$

The sequence (6.7) arises from the sequence (6.2) by *projectivization*. The parameter k (or rather $2k$) is encoded in the Poisson structure. Let $\mathcal{O}(k)$ be the k'th power of the hyperplane bundle on $\mathbb{C}\mathrm{P}^d$, let

$$\iota_{Q_s} \colon Q_s \longrightarrow Q_\ell \cong \mathbb{C}\mathrm{P}^d$$

be the inclusion, and let $\mathcal{O}_{Q_s}(k) = \iota^*_{Q_s}\mathcal{O}(k)$. The quantum *Hilbert* space amounts now to the space of holomorphic sections of $\iota^*_{Q_s}\mathcal{O}(k)$, and the resulting *costratified quantum Hilbert space* has the form

$$\Gamma^{\mathrm{hol}}(\mathcal{O}_{Q_1}(k)) \longleftarrow \ldots \longleftarrow \Gamma^{\mathrm{hol}}(\mathcal{O}_{Q_s}(k)).$$

Each vector space $\Gamma^{\mathrm{hol}}(\mathcal{O}_{Q_{s'}}(k))$ $(1 \leq s' \leq s)$ is a finite-dimensional representation space for the quantizable observables in $C^\infty(Q_s)$, in particular, a $\mathfrak{u}(\ell)$-representation, and this representation integrates to a $\mathrm{U}(\ell)$-representation, and each arrow is a morphism of representations; similarly as before, these arrows are just restriction maps.

We will now give a description of the decomposition of the space

$$\Gamma^{\mathrm{hol}}(\mathcal{O}_{Q_\ell}(k)) = S^k_{\mathbb{C}}[\mathfrak{p}^*]$$

of homogeneous degree k polynomial functions on $\mathfrak{p} = S^2_{\mathbb{C}}[\mathbb{C}^\ell]$ into its *irreducible* $\mathrm{U}(\ell)$-representations in terms of highest weight vectors. To this end we note that coordinates $x_1, \ldots, x_\ell$ on $\mathbb{C}^\ell$ give rise to coordinates of the kind $\{x_{i,j} = x_{j,i};\ 1 \leq i, j \leq \ell\}$ on $S^2_{\mathbb{C}}[\mathbb{C}^\ell]$, and the determinants

$$\delta_1 = x_{1,1},\ \delta_2 = \begin{vmatrix} x_{1,1} & x_{1,2} \\ x_{1,2} & x_{2,2} \end{vmatrix},\ \delta_3 = \begin{vmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{1,2} & x_{2,2} & x_{2,3} \\ x_{1,3} & x_{2,3} & x_{3,3} \end{vmatrix},\ \text{etc.}$$

are highest weight vectors for certain $\mathrm{U}(\ell)$-representations. For $1 \leq s \leq r$ and $k \geq 1$, the $\mathrm{U}(\ell)$-representation $\Gamma^{\mathrm{hol}}(\mathcal{O}_{Q_s}(k))$ is the sum of the irreducible representations having as highest weight vectors the monomials

$$\delta_1^{\alpha}\delta_2^{\beta}\dots\delta_s^{\gamma}, \quad \alpha + 2\beta + \dots + s\gamma = k,$$

and the restriction morphism

$$\Gamma^{\mathrm{hol}}(\mathcal{O}_{Q_s}(k)) \longrightarrow \Gamma^{\mathrm{hol}}(\mathcal{O}_{Q_{s-1}}(k))$$

has the span of the representations involving δ_s explicitly as its kernel and, restricted to the span of those irreducible representations which do *not* involve δ_s, this morphism is an isomorphism.

This situation may be interpreted in the following fashion: The composite

$$\mu_{2k}\colon \overline{\mathcal{O}}_s \subseteq \mathfrak{sp}(\ell, \mathbb{R}) \cong \mathfrak{sp}(\ell, \mathbb{R})^* \longrightarrow \mathfrak{u}(\ell)^*$$

is a singular momentum mapping for the $\mathrm{U}(\ell)$-action on $\overline{\mathcal{O}}_s$; actually, the adjoint $\mathfrak{u}(\ell) \to C^{\infty}(\overline{\mathcal{O}}_s)$ of μ^{2k} amounts to the composite of (6.4) with the inclusion of $\mathfrak{u}(\ell)$ into $\mathfrak{sp}(\ell, \mathbb{R})$. The *irreducible* $\mathrm{U}(\ell)$*-representations which correspond to the coadjoint orbits in the image*

$$\mu_{2k}(O_{s'} \setminus O_{s'-1}) \subseteq \mathfrak{u}(\ell)^*$$

of the stratum $O_{s'} \setminus O_{s'-1}$ $(1 \leq s' \leq s)$ *are precisely the irreducible representations having as highest weight vectors the monomials*

$$\delta_1^{\alpha}\delta_2^{\beta}\dots\delta_{s'}^{\gamma} \quad (\alpha + 2\beta + \dots + s'\gamma = k)$$

involving $\delta_{s'}$ *explicitly, i. e. with* $\gamma \geq 1$.

7 Applications and outlook

In [5], Atiyah and Bott raised the issue of *determining the singularities* of moduli spaces of semistable holomorphic vector bundles or, more generally, of moduli spaces of semistable principal bundles on a non-singular complex projective curve. The complex analytic stratified Kähler structure which we isolated on a moduli space of this kind, as explained in Example 2 above, actually determines the singularity structure; in particular, near any point, the structure may be understood in terms of a suitable local model. The appropriate notion of singularity is that of singularity in the sense of stratified Kähler spaces; this notion depends on the entire structure, not just on the

complex analytic structure. Indeed, the examples spelled out above (the exotic plane with a single vertex, the exotic plane with two vertices, the 3-dimensional complex projective space with the Kummer surface as singular locus, etc.) show that a point of a stratified Kähler space may well be a singular point without being a complex analytic singularity.

A number of applications of the theory of stratified Kähler spaces have already been mentioned. Using the approach to lattice gauge theory in [14], we intend to develop elsewhere a rigorous approach to the quantization of certain lattice gauge theories by means of the Kähler quantization scheme for complex analytic stratified Kähler spaces explained in the present paper. In collaboration with M. Schmidt and G. Rudolph, we plan to apply this scheme to situations of the kind explored in [21] –[23] and to compare it with the approach to quantization in these papers. Constrained quantum systems occur in molecular mechanics as well, see e. g. [30] and the references there. Perhaps the Kähler quantization scheme for complex analytic stratified Kähler spaces will shed new light on these quantum systems.

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Université des Sciences et Technologies de Lille, UFR de Mathématiques, CNRS-UMR 8524
59655 VILLENEUVE D'ASCQ, Cédex, France
and
Institute for Theoretical Physics, Universität Leipzig
04109 LEIPZIG, Germany
Johannes.Huebschmann@math.univ-lille1.fr